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Demonstrating mathematics learning as the emergence of eye–hand dynamic equilibrium

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Abstract
This paper combines recent developments in theories of knowledge (complex dynamic systems), technologies (embodied interactions), and research tools (multimodal data collection and analysis) to offer new insights into how conceptual mathematical understanding can emerge. A complex dynamic system view models mathematics learning in terms of a multimodal agent who encounters a set of task constraints. The learning process in this context includes destabilizing a systemic configuration (for example, coordination of eye and hand movements) and forming new dynamic stability adapted to the task constraints. To test this model empirically, we applied a method developed to study complex systems, recurrence quantification analysis (RQA), to investigate students' eye–hand dynamics during a touchscreen mathematics activity for the concept of proportionality. We found that across participants ($n = 32$), fluently coordinated hand-movement solutions coincided with more stable and predictable gaze patterns. We present a case study of a prototypical participant's hand–eye RQA and audio–video data to show how the student's cognitive system transitioned out of prior coordination reflective of additive thinking into a new coordination that can ground multiplicative thinking. These findings constitute empirical substantiation in mathematics education research for cognition as a complex system transitioning among dynamic equilibria.

Keywords Complex dynamic systems · Multimodal mathematics learning · Proportion · Coordination dynamics · Learning analytics

1 Introduction

Human development is a nonlinear process that occurs in, for, and through multimodal interaction with the environment (Adolph et al., 2018; Allen & Bickhard, 2013; Kelso, 2016; Spencer et al., 2012). For educational researchers who follow this nonlinear stance,

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a hard nut to crack is how multimodal interaction with the environment could give rise to higher-order mental activity, such as understanding mathematical concepts (Hutto et al., 2015). The alleged multimodal grounding of cultural knowledge has been increasingly considered through theories loosely referred to as embodied cognition (Anderson, 2003; Barsalou, 2010; Galetzka, 2017; Glenberg, 2010; Varela et al., 1991; Wilson, 2002). In turn, educational researchers are evaluating the implications of embodied approaches for advancing the theory and practice of teaching and learning mathematics (Abrahamson & Bakker, 2016; Dackermann et al., 2017; de Freitas & Sinclair, 2013; Goldenberg et al., 2008; Radford, 2009).

Embodiment theories have been drawn on to inform the design of pedagogical activities and digital environments, wherein embodied interaction undergirds conceptual learning (Abrahamson & Trninic, 2011; Lindgren and Johnson-Glenberg, 2013). Students learn mathematics in these contexts by solving motor-control problems with virtual objects linked to symbolic expressions of concepts. For example, students learn to coordinate one hand's movement on the unit circle with the other's movement on the Cartesian plane (sin graph) (Alberto et al., 2019). The technological design builds on the embodied-interaction approach to technology development (e.g., Dourish, 2001) that foregrounds sensorimotor competence as the epistemic substance of disciplinary know-how. Per enactivism (Varela et al., 1991), the epistemic key to developing sensorimotor know-how in a discipline is coming to perceive the domain of scrutiny in a new way. New cognitive structures emerge from repeated patterns of visual attention that guide the sensorimotor enactment of assigned movement tasks. Cognitive psychologists of movement have demonstrated the centrality of visual perception in performing motor-control tasks (Mechsner, 2003, 2004; Muraoka et al., 2016). This line of research has led to theorizing the role of movement in computer-mediated mathematics learning (Abrahamson & Bakker, 2016).

Eye trackers are research instruments for measuring people's eye positions and eye movements. Eye trackers provide data to inform investigations into cognitive processes involving visual information, such as mathematics learning (Schindler & Lilienthal, 2019; Strohmaier, 2020). Specifically, eye-tracking methodologies support an empirical evaluation of the enactivist view on the multi-modal grounding of mathematical concepts in cognitive structures that emerge as students learn to enact movements that solve embodied-interaction problems (Shayan et al., 2015). Eye-tracking data, triangulated with video capture of students' verbal and gestural utterances, can determine how students' visual orientation guides their enactment of new manual coordination (Abrahamson et al., 2015; Duijzer et al., 2017; Shvarts & Abrahamson, 2019). Tancredi et al. (2021) analyzed students' bimanual actions as they solved mathematical motor-control problems. In that study, we demonstrated how learners' hands self-organized in response to task constraints to form dynamically stable movement-pattern solutions. Tancredi et al. (2021) were arguably the first to document the sensorimotor emergence of mathematical enactment quantitatively and, as such, to lend empirical credence to models of conceptual learning as a complex dynamic system (CDS) in flux (see Sect. 2.1) (cf. Thelen and Smith, 2006).

This paper further demonstrates the emergence of visual patterns guiding the enactment of mathematical movement by integrating contemporaneous eye-gaze movement patterns on top of the bimanual coordination found in Tancredi et al. (2021). It uses a CDS view of human cognition and enactivist accounts of mathematical phenomenology as theoretical lenses on mathematics learning (Pirie & Kieren, 1989; Reid, 2014; Steffe & Kieren, 1994). Using mixed analyses, we argue and demonstrate that the emergence of bimanual movement patterns depends on the contemporaneous and dynamically interleaved emerging of visual patterns, as predicted by enactivist theory.

2 Theoretical framework and implementation context

2.1 Complex dynamic systems

Complex dynamic systems are composed of components that constitute a sustained functional structure. They are dynamic because they are in flux (even when appearing static). They are nonlinear in that their change is a property of a system where the output is not directly proportional to the input. For example, small changes in the input can lead to significant changes in the output or vice versa. A system is complex when the components are interconnected through nonlinear relationships, giving rise to a new ontological entity. The CDS approach investigates functional networks of distributed components as they enter and exit stable coordination in response to perturbations (Kelso, 2010; Koopmans, 2020; Kostrubiec et al., 2012; Scheffer et al., 2009). For example, as a horse increases its locomotion efforts (endogenous perturbation), new coordination among its legs self-organizes and then dissolves, transitioning across four dynamically stable phases or attractors: walk, trot, canter, and gallop (Schöner et al., 1990; Thelen & Smith, 2006). When a CDS responds to changes in some constraint, we refer to that constraint as the system's control parameter (analogous to an independent variable in linear systems). For example, imagine a horse on a treadmill—the (exogenous) control parameter would be the treadmill's speed that gives rise to the re-coordination of its gait organization. We measure the change in a CDS's phase transitions through an order parameter (analogous to a dependent variable). The gait of a horse would be gauged by order parameters such as the phase synchrony or rhythm of its four legs' contact with the treadmill.

2.2 Complex dynamic systems and human learning

Human cognition can be seen as a complex dynamic system constituted of distinguishable components (e.g., Anderson et al., 2012; Kelso, 2010; Stephen & Dixon, 2009). Cognitive development can be modeled as assemblages of components in flux that bind into and out of functionally effective configurations (Thelen & Smith, 2006). Modeling human cognition as a complex system transitioning between equilibrium phases was imported to psychology by neuroscientists and ecological anthropologists inspired by the systemic paradigm of cybernetics (Bateson, 1972). Researchers have used CDS to model manual dexterity (Kelso, 1984), infants' development of agency (Kelso, 2016), toddlers' performance on Piagetian tasks (Wilson & Golonka, 2013), and adults' solving of logical problems (Stephen & Dixon, 2009). In the early twentieth century, cybernetics scholarship became known in the natural sciences and later spread through the social sciences, cognitive-developmental psychology, and educational research (Clancey, 2008). Some of these interdisciplinary scions of cybernetics are only of late being retroactively appreciated as tributaries to what is becoming a new paradigm of human cognition, development, and learning (Nagataki & Hirose, 2007).

Coordination dynamics provides a unifying framework for understanding how components of biological constitution come together across scales (e.g., neural to behavioral to social) (Kelso, 1995). Coordination dynamics studies identify critical task-dependent order among modalities, the nonlinear relations among modalities that give rise to their coordination, and the rules that govern the stability and change of patterns (Kostrubiec et al., 2012; Stephen & Dixon, 2009). From a coordination dynamics perspective, instruction is creating *fields of promoted action* (Reed & Bril, 1996)—perturbations that stimulate learners' responses congruent with culturally desired skills. To do so, instructors introduce, remove, or modify a set of tasks and environmental constraints—either directly or through augmenting the information the learner receives—and the learner devises means of re-coordinating their motor actions to enact movements that perform the task (Chow et al., 2021; Newell & Ranganathan, 2010). Such constraint-based pedagogy seeks to create opportunities for competitive athletes to discover their own optimal, often idiosyncratic, motor-action solutions to these enactmen problems (Lee et al., 2014; Liao & Masters, 2001).

An attractor is a fixed point in a system's trajectory toward which the system tends to evolve. It represents a potential goal state for the actor and the learning system. By analogy, an attractor can be seen as a marble rolling in a smooth, rounded basin that will always come to rest at the lowest point, in the bottom center of the bowl; this final state of position and motionlessness is a point attractor. A basin of attraction is a set of points from which a dynamical system spontaneously moves to a particular attractor. In the following sections, we describe how the learner can develop coordination between gaze and hand movement using these concepts borrowed from CDS methodology.

2.3 An empirical context: the Mathematics Imagery Trainer for proportion

The Mathematics Imagery Trainer (henceforward, the Trainer) is a learning-environment architecture that exemplifies the action-based embodied design framework (Abrahamson et al., 2014; Alberto et al., 2021; Shvarts & van Helden, 2021). The Trainers are embodied-interaction design architectures for enactive learning of mathematical concepts through solving motor-control problems (Reinholz et al., 2010) in dedicated environments dubbed instrumented fields of promoted action (Abrahamson & Trninic, 2015). Trainer tasks require learners to discover and enact a movement form that generates a particular goal state of the interactive system. To date, Trainers have been used to foster and study the Cartesian field (Abrahamson & Bakker, 2016), trigonometry (Alberto et al., 2019), and parabolas (Shvarts & Abrahamson, 2019).

The particular Trainer task in this study focuses on proportion (Abrahamson et al., 2014). The design of the Trainer for proportion responds to the well-documented difficulty students incur with rational numbers, such as a/b fractions or a:b ratios (e.g., Lamon, 2007). In studies of conceptual transitions from additive to multiplicative reasoning, students' solutions to Trainer problems typically begin by attempting to maintain a constant additive relation between the magnitudes in question by moving both hands at the same speed (Fig. 1b, c). A paradigmatic interaction sequence of Grades 4–6 study participants toward discovering how to enact a "green" movement form (Fig. 1) transpires as follows: (a) while exploring, the student first positions the hands

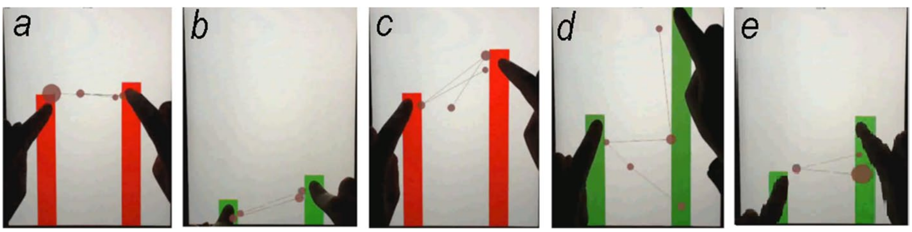

Fig. 1 Mathematics Imagery Trainer for proportion. Set at a 1:2 ratio, the bars turn green only when the right-hand bar is twice as tall as the left-hand bar

incorrectly in a variety of "red" locations; (b) stumbles upon a "green" position; (c) raises their hands, maintaining a fixed interval between them, resulting in red feedback; (d) corrects back to a "green" position; and (e) eventually figures out how to move while keeping the bars green.

This initial phase is Exploration: manipulating the bars in different ways to find what makes them green. When a learner successfully elicits green feedback, this marks the Discovery phase—the learner can begin to form and investigate ideas about how green is generated, identifying discrete green positions. Through a trial-and-error exploration process, students realize that the critical relation between the magnitudes is not additive. This leads to the third phase, Fluency, wherein learners successfully organize themselves to move the bars continuously while keeping them green. They develop a new bimanual coordination dynamic that attends to a new invariant property that could be described as multiplicative (Abrahamson & Abdu, 2020). Indeed, the students articulate their movement strategy as maintaining an invariant multiplicative relation, such as keeping the left bar half as high as the right bar. Analogous to learning other movements, such as dancing a pirouette, fluency is marked by smoothness. Our analyses will focus on these initial three phases. An instructor then introduces supplemental mathematical instruments into the activity space, such as grids and numerals. When participants incorporate these symbolic artifacts into their enactment, they transition into new ways of moving, thinking, talking, and representing the mathematical concept of proportion (Abdu et al., 2021; Abrahamson et al., 2011; Bongers, 2020). Mastering the application of additive vs. multiplicative reasoning has been repeatedly demonstrated as key to learning rational numbers (e.g., Van Dooren et al., 2010). Significantly for our thesis, solving a Trainer movement problem means that the complex dynamic multimodal system arrives at a new equilibrium.

2.4 Perceptual solutions to movement problems

Enactivist theory predicts that learning to move in a new way is contingent on learning to perceive the environment in a new way ("perceptually guided action," Varela et al., 1991, p. 173). In line with the enactivist hypothesis, students who solved the Trainer task in Abrahamson et al. (2014) referred to new "things" they saw as their practical means of solving the task. They attended to the spatial interval between the top and middle of the right bar—a negative space in one's visual field that came forth as a phenomenal object handled to maintain the bars green while moving the hands. The emergence of these visual structures coincided with improved performance (Abrahamson & Trninic, 2011; Reinholz et al., 2010). Later studies applied eye tracking to empirically demonstrate the emergence of these dynamically stable, goal-oriented visual structures (Duijzer et al., 2017; Shvarts & Abrahamson, 2019). Abrahamson and Sánchez-García (2016) used the term *attentional anchors* to signify perceptual solutions to the problem of coordinating complex situated movements. The attentional anchor emerges through goal-oriented iterated exploration, discovery, adaptation, and refinement, until it coalesces as mediating the enactment of the movement form.

Per the CDS perspective, attentional anchors are self-imposed constraints that reduce the degrees of freedom of the problem space (Abrahamson & Abdu, 2020; Savelsbergh et al., 2004). The mathematical function underlying the task demand, such as a 1:2 ratio, is a control parameter because it is activated to perturb the student's business-as-usual additive movement schemes (fixed distance between hands, Fig. 1b, c) and nudges him to precipitate toward proportional movement schemes (changing distance between hands, Fig. 1d, e). The activity's order parameter is implicated via the green feedback that acts as a proxy, indicating that the student has reconfigured their enactment scheme.

2.5 The current study

This paper uses quantitative means to demonstrate the microprocesses of multimodal mathematics learning as a complex eye–hand system's reorganization toward equilibrium. New methodologies for gathering, analyzing, and modeling learners' gazes enable a direct look at the microprocesses of mathematics learning (Blikstein & Worsley, 2016; Noroozi et al., 2019). The methodology used in this study, recurrence quantification analysis (RQA), is uniquely geared to quantify and model microprocesses in terms of repetition, stability, and predictability (Stephen & Dixon, 2009). RQA can model individuals' sensorimotor manifold's complex and dynamic evolution as they engage in embodied-interaction problem-solving tasks within concept-oriented educational environments. In Tancredi et al., (2021, 2022), we used RQA to compare the nonlinear dynamics of participants' bimanual coordination across different stages of interaction with a Trainer activity, offering a detailed account of task solution as bimanual movement destabilizing and then reconfiguring into task-effective coordination.

We are specifically interested in eye-gaze orientation developments when students interact with Trainer tasks. We attempt to answer two research questions. First, how does students' eye–hand coordination change as they solve embodied mathematics Trainer tasks, and how are these changes consistent with the nonlinear phase transition of complex dynamic systems? Second, how do complex dynamic eye–hand systems shift into a new dynamic equilibrium when students solve embodied mathematics Trainer tasks? We set up a congruent research design that would enable us to identify how the students' visual constructions and their motor performance reach task-effective coordination.

3 Methodology

3.1 Participants

Forty-five 5th and 6th students from the Netherlands participated in a study reported in previous publications (Abrahamson et al., 2015; Duijzer et al., 2017; Tancredi et al., 2021, 2022). Data were removed due to technical issues (8 participants) and not meeting the experiments' stage threshold (5 participants, see paragraph 3.4). Ultimately, touchscreen hand position coordinates, eye-gaze coordinates, and audio–video data of manual actions and verbal-gestural utterances by the remaining 32 participants were included in the analysis.

3.2 Task

The task was implemented using an Apple iPad Air touchscreen (resolution 1536×2048 p.; diagonal 24.6 cm; refresh rate 120 Hz) in the Trainer application (Duijzer et al., 2017). Individual students participated in approximately 1hour sessions in a private room at school. Participants sat in front of the tablet. Each session started with a task explanation read by a student: "You have to move the bars up and down and find the green bars. Try to keep the bars green while moving them." Students first explored the problem space with minimal intervention. Once participants found a green position, the researcher prompted them to find more while thinking aloud using pre-formulated questions such as "Can you find more greens?" and "Could you tell me what you are doing right now?" Sessions ended when the students successfully moved the bars simultaneously from the bottom to the top while keeping the bars green (mean duration of 7:11 min).

3.3 Data collection and preprocessing

The research team collected eye-gaze and both hands' *x*- and *y*-locations. Eye-gaze location was collected with a screen-based Tobii X2-30 eye-tracker (30 Hz, S.D. = 2 Hz), which allows head movement. The accuracy of the Tobii X2-30 eye tracker under similar conditions is approximately 2.46°, and its precision is approximately 1.91° (Clemotte et al., 2014). Participants were asked to keep looking at the tablet screen. Calibration included looking at the four corners of the tablet screen and its middle. No threshold was defined for the calibration accuracy. Preprocessing raw gaze data was done first with Tobii's fixation filter. After correcting for missing data points (below 100 ms), the filter identifies fixation points in at least five gaze points grouped within a 35-pixel radius. In addition to eye-tracking data, the Trainer app recorded touchscreen positions for each finger throughout the process (a varied sampling frequency of ~ 50–120 Hz).

To align the sample rates for fingers and gaze data, we downsampled both to a rate of 10 Hz (see Tancredi et al., 2021). We programmed a Python script to make these alignments automatically and produced synthetic videos of hands and eye-gaze locations over time. Our research team then watched these movies to verify spatial alignment using videos containing eye-gaze data overlaid atop the video (Fig. 1) and transcripts of the interaction translated from Dutch to English (produced for Duijzer et al., 2017). Unaligned data files ($N = 22$ files) were then adjusted manually by Authors 1 and 2.

3.4 Coding for the three stages

We split each participant's time series data according to typical stages in solving the Trainer task: Exploration (before discovering any green), Discovery (looking for new green locations after finding a first one), and Fluency (moving the hands together in green). Using an R script, we detected these stages automatically. We used a proxy variable for "green location," coded 1 if the left-hand *y* coordinate divided by the right-hand *y* coordinate was between 0.4 and 0.6 for a given interval of 100 ms, and coded 0 if not. We calculated a two-sided rolling average of the green location variable with a window size of 20 s. The Exploration stage began with the first dataset entry and ended when a participant reached the first milestone: finding and maintaining a green location for at least 10 of 20 consecutive seconds. We set the window size to 20 s to ensure the participant engaged with the green location for a substantial period rather than passing through it briefly without returning. Reaching this milestone marked the beginning of the Discovery phase. We used another proxy variable, moving-in-green, to set the next transition. Moving-in-green was set to 1 when, in the last 100 ms, the left and right fingers changed positions, and the green location variable was equal to 1. We ran a two-sided rolling average of this variable with a 10-s window size. The Fluency stage began when the rolling average over a 10-s span reached a value higher than 80% of that individual's maximum rolling average value. This threshold was set to mark moving in green close to participants' personal best. Fluency's window size was shorter than Discovery because it identified a local dynamic of high-level performance rather than the general dynamic of engaging with greens frequently.

Authors 1 and 2 hand-coded the transition points independently to validate the automatic coding by visually evaluating graphs of a series of right- and left-hand heights using working definitions of the transitions between stages. We identified discrepancies in transition point locations larger than 30 s among the automatic and human coders. Based on these values, we reached an agreement, revising nine automated coding values (5 for Discovery and 4 for Fluency). Lastly, participants with stages shorter than 7 s were omitted from the dataset (3 for Exploration, 1 for Discovery, and 1 for Fluency), yielding a total dataset of 32 participants.

3.5 Coding for areas of interest

We were interested in changes in eye-gaze patterns over time vis-à-vis the hands' positions and ratios. For example, looking at a location in the upper right portion of the screen takes on a different meaning if the right hand is in that location than if the hands are at the screen's bottom. We adapted Duijzer et al.'s (2017) areas of interest (AoI), subdividing the space and coding eye-gaze according to these AoI for each 100 ms sample. To focus the analysis on the enactment space, we identified the borders of the tablet screen. We defined a proxy range variable that equals the accumulation of the two hands' *x*-location medians divided by 8. The left and right borders of the screen (see Fig. 2) were calculated for each participant as the median of their left hand's *x* location minus the range variable and the median of their right hand's *x* location plus the range

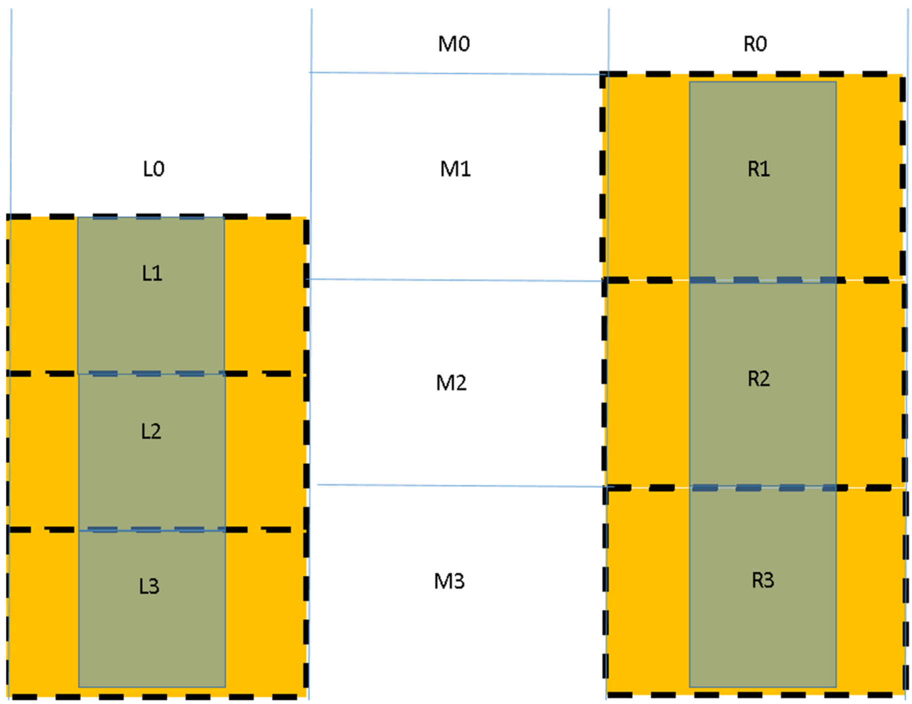

Fig. 2 Coding scheme for areas of interest

variable. The screen was then partitioned into three columns of equal width (~512 p each): left for the left bar, right for the right bar, and middle for eye-gaze between the bars.

The vertical partitioning of the screen was done separately for each column. We partitioned the left and right columns to four AoIs with an R code. For every 100 ms timestamp, we split the left and right bars into three equal AoIs, following Duijzer et al. (2017). To account for the possibility that participants' eye-gaze would be slightly above the fingers, the top bar AoI was extended upwards by a factor of 1.1 spatial variance in eye-gaze (AoI L1 and R1 in Fig. 2). We defined the region between the top of the screen and the tops of AoI L1 and R1 and a third AoI between these columns. The middle column was calculated as follows: when the right bar was taller than the left bar, the vertical allocation was similar to the vertical allocation of the right bar; when the left bar was taller than the right bar (does not appear in Fig. 2), the middle column was split into three AoIs defined by the height of the right bar, the height of the left bar, and the top of the screen (M1, M2, and M3 in Fig. 2).

3.6 Recurrence quantification analysis (RQA)

RQA is a nonlinear method that captures the structure of variability in a dynamic system over time. Marwan et al. (2007) applied RQA to study dynamic systems across disciplines, such as using participants' movements to predict moments of insight in problem solving (Stephen & Dixon, 2009) and model their scientific beliefs (Fleuchaus et al., 2020). In Tancredi et al. (2021), we studied the development of dynamic bimanual stability as learners developed new proto-concepts of multiplicative reasoning. Here, we recruit this method to study the structure of learners' eye-gaze patterns as they learn to coordinate their hands in a new way. We use categorical auto-RQA—a type of RQA that examines recurring events within a single categorical time series by comparing each system's state over time to every other by creating a recurrence plot (see Fig. 3). In Fig. 3, we present an auto-RQA plot where a focal time series is placed on both the *x*- and *y*-axes. Every point is compared to every other. For example, the first AoI in the time series (L1, see Fig. 2 for instance), appearing in the bottom left corner, is compared to each of the seven values along the plot's horizontal and vertical axes. When it appears again (the fourth entry), a point appears on the plot (here, in green).

Different metrics of the resulting plot are quantified to reflect various aspects of the time series. For example, the total number of points on the plot demonstrates the overall recurrence in the time series; the average vertical line length (such as the height of the yellow box resulting from looking at area R0 for 300 ms) reflects the duration of persistent positions.

We used the crqa package in R (Coco & Dale, 2014) to conduct a categorical auto-RQA for eye-gaze AoI for each of the three bimanual problem-solving stages. We used conventional categorical RQA parameters (radius = 0.001, delay = 1) and an embedding dimension of 5, ignoring recurrent sections shorter than 500 ms. We chose five RQA metrics (Marwan et al., 2007)—recurrence rate, percent determinism, mean line length, normalized entropy, and trapping time-to characterize gaze dynamics across the three stages. The recurrence rate (scale = 0–100) measures repetition levels, calculated as the percentage of recurring points in the RQA plot. Percent determinism (0–100 scale) reflects the system's predictability—calculated as the percentage of plot points on diagonal lines longer than 1 point. Mean line length (scale = 0–½ time-series

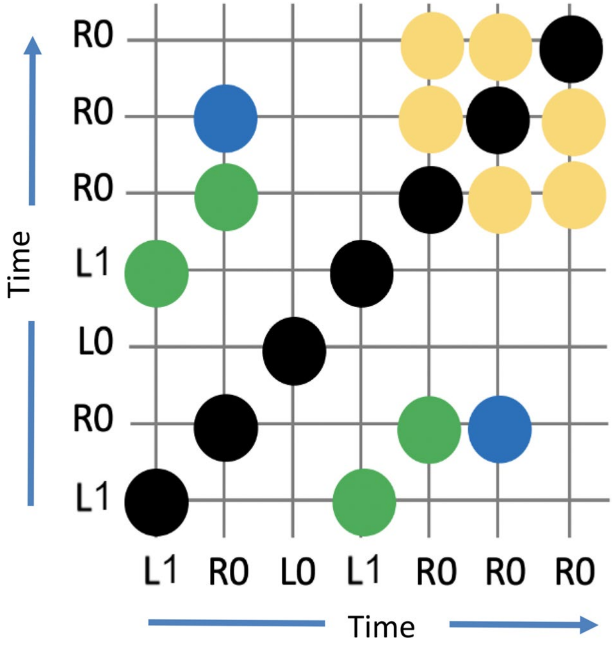

Fig. 3 An example of a categorical auto-recurrence plot. Note: the coloring here is used for illustration. Typically, RQA plots for large datasets and recurrence variations are monochromatic (Fig. 5). Note also that this plot was created for illustrative purposes and is not grounded in actual data

length) reflects the system's stability—calculated as the average length of diagonal lines longer than 1 point on the recurrence plot. Shannon's normalized entropy (scale = 0–1) reflects the level of disorder in the system, calculated as the distribution of lengths of the lines in the plot normalized by the total number of lines in the recurrence plot. A decrease in entropy reflects an increase in the system's order. Trapping time (scale = 0–½ time-series length) demonstrates the system's consistency in terms of the average duration of a connected state—calculated as the average of the vertical lines' lengths in the recurrence plot. To test for the main effects in the RQA metrics between stages (see Table 2), we conducted a Wilks-Lambda repeated-measures ANOVA ($n = 32$). In the post hoc analysis, we used Fisher's least significant difference test to compare the three stages.

3.7 Choosing a case study

We analyzed one participant's video data and RQA metrics (Jan, pseudonym). We chose this participant because his learning process can be comprehensively described and based on its overall similarity to the cross-participant main effect (see Tables 1 and 2): Jan is representative in all but trapping time in the Exploration stage, which went up rather than of down. We chose five critical moments to analyze qualitatively based on a triangulation of: (a) video data, including eye-gaze; (b) an RQA plot of Jan's performance throughout the activity (Fig. 5, up); and (c) hand locations over time (Fig. 5, bottom). Based on internal discussions among the authors, we refined our analysis of the critical moments to develop a representation of phase transitions of a complex dynamic eye–hand system (Figs. 6, 7, 8, 9 and 10).

4 Findings

4.1 Quantitative changes in RQA metrics

All RQA metrics, except entropy, increased in the Fluency stage (Fig. 4). For all, a significant main effect ($p < 0.05$) was recorded (Table 1). Recurrence rate and percent determinism were significantly different, higher only in the Fluency stage, implicating increased eye-gaze AoI repetition and predictability. The mean line length and trapping time dropped from Exploration and Discovery. They increased from Discovery to Fluency, implicating a resemblance between the Exploration and Fluency stages regarding the stability and consistency of the systems. Entropy was highest in the Exploration stage and significantly lower in the Discovery and Fluency stages, indicating a decrease and a non-significant

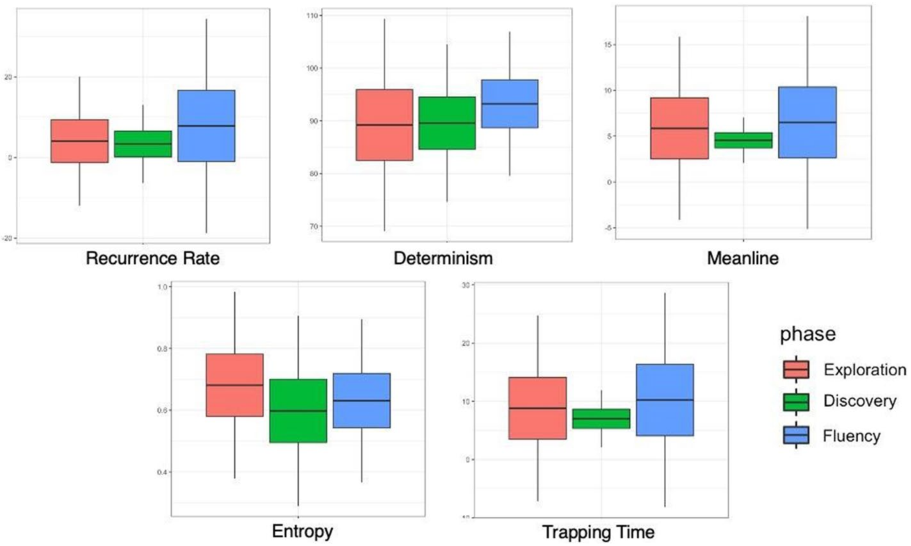

Fig. 4 Boxplot for the five RQA metrics over the three study stages ($n = 32$). Center line, mean; Box, 1 SD; whiskers, 3SDs

Table 1 Estimated means and standard deviations of RQA metrics across participants ($n = 32$) in the three study stages

Stage	Exploration		Discovery		Fluency
Recurrence rate*	4.04 (5.34)		3.33 (3.23)	<	7.82 (8.85)
Determinism (%) *	89.21 (6.71)		89.55 (4.98)	<	93.22 (4.55)
Mean line length*	5.85 (3.34)	>	4.54 (.83)	<	6.49 (3.87)
Normalized entropy*	0.68 (.10)	>	0.60 (.11)		0.63 (0.09)
Trapping time*	8.80 (5.31)	>	6.99 (1.64)	<	10.22 (6.14)

The main effect was observed for all variables. We present the significance of the difference between Exploration and Discovery (Column 3) and between Discovery and Fluency (Column 5). Exploration and Fluency are significantly different for recurrence rate, determinism, and normalized entropy increase in the system's disorder (steady states were more disordered). Recurrence rate, mean line length, and trapping time also showed lower variance in the Discovery phase than in the other two phases.

4.2 Case study: phase transition

To contextualize the main mathematics learning effects, we illustrate and analyze the learning sequence of one participant (Jan, pseudonym). For Jan, increases in the Recurrence rate, determinism, mean line, and trapping time metrics indicate increases in repetition, predictability, stability, and duration of continuous focus of the eye–hand system in the Fluency stage (Table 2). Entropy increased during the Fluency stage, indicating an increase in the system's variability of recurring coordinations.

The RQA plot of Jan's solution process (Fig. 5) demonstrates a phase transition between two distinct equilibria. The time series of gaze AoIs forms both the *x*- and *y*-axis of the recurrence plot, and local dynamics at each moment in the time series run along the diagonal from the bottom left of the RQA plot to the upper right (Fig. 5, top). The plot begins in the lower-left corner with a dense blue square, demonstrating the high consistency of the CDS at this stage. This initial gaze stability corresponds to the first segment of bimanual dynamics shown in the lower graph (Fig. 5). Around minute 1:40, Jan finds green (the lower graph in Fig. 5, ~ 1:40 to ~ 3:40). Moving up and right along the recurrence plot diagonal, we observe how eye-gaze dynamics change after this green feedback, showing a new and somewhat less stable structure. Progressively, as Jan finds more greens (~ 3:40 to ~ 10:10), his recurrence plot shows increasingly sparse and less organized structures on the plot as the system fluctuates out of dynamic equilibrium. Finally, as the hands move in green, the system reaches equilibrium around the (10:20) mark, demonstrated as a dense square representing high recurrence levels. In Fig. 5, we mark five key moments (a–e) in which this CDS fluctuates out of dynamic equilibrium (a–c) and then back into (d, e) dynamic equilibrium in the interaction with the Trainer and the teacher.

In Fig. 5, the beginnings of the Discovery and Fluency stages are signified with dotted and solid lines (6:01 and 9:29, respectively). Key moment markers (a–e) indicate changes in movement patterns (observable in the bottom figure). The density of blue dots in the squares marked on the RQA plot's main diagonal represent eye–hand coordination: two solid squares that represent high coordination, then a fluctuation (a) accompanied by lowering density squares (b to c to d) and the emergence of new coordination shown as a higher density square (e).

We now detail the five key moments in Fig. 5. We chose five corresponding video snapshats (Figs. 6–10) to exemplify eye-gaze patterns around those critical moments. Jan tried to solve the Trainer's problem while an experimenter prompted him to speak aloud (see Subsection 3.2). Jan's and the researcher's words are highlighted in italics, and […] represents omission, for narrative brevity.

Table 2 Jan's auto-RQA metrics of the Exploration (00:00–6:01), Discovery (6:01–9:29), and Fluency (9:29–12:11) stages

Stage	Exploration	Discovery	Fluency
Recurrence rate	1.03	0.91	2.59
Determinism	77.51	80.96	91.16
Mean line	3.15	3.59	5.33
Entropy	0.50	0.49	0.59
Trapping time	4.41	5.38	8.59

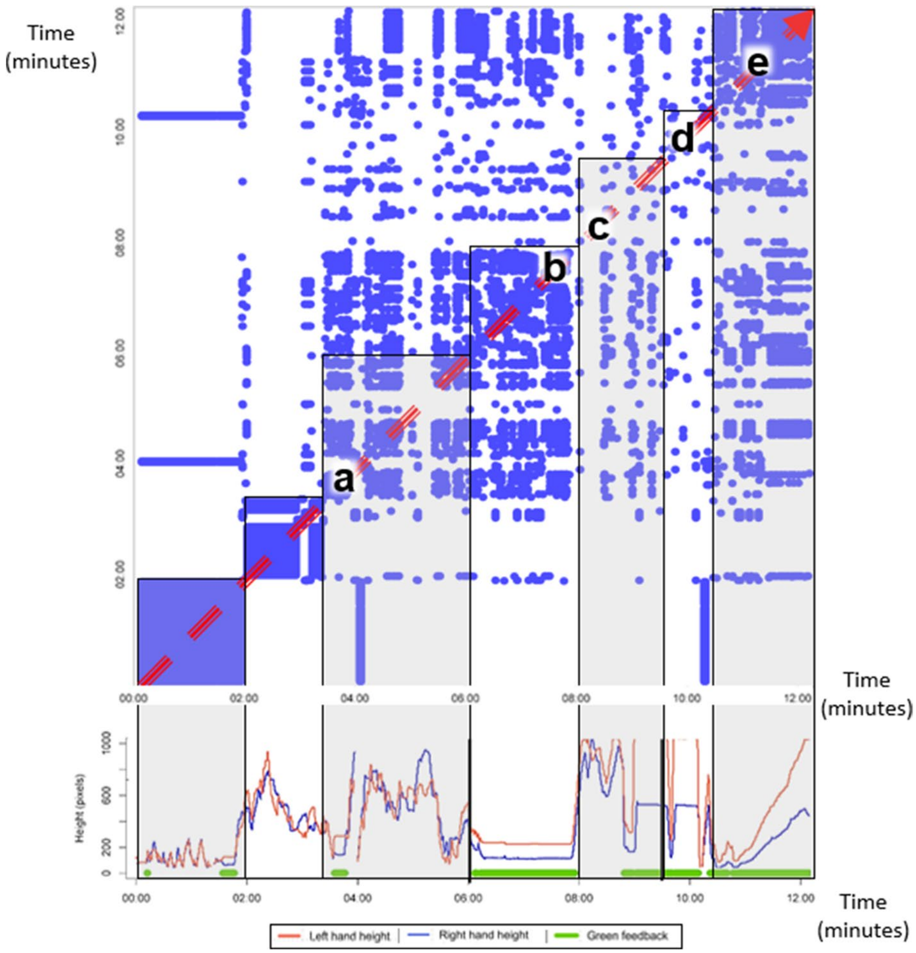

Fig. 5 Auto RQA plots Jan's eye-gaze areas of interest (top) and his hands' locations throughout the solution process and green feedback (bottom)

The session began with the following encounter: the researcher asked Jan to "find out when the bars turn green," stressing that he "can move [his] fingers" (00:00–00:12). After a minute and a half, during which Jan reaches a green position for a brief period, the researcher elicits Jan's thoughts, and Jan explains: "[…] I thought they [the bars] were the same for a while, then they turned green, but that is not the case." That is, Jan expresses the hypothesis that green locations requires the two bars to be at the same height. The RQA plot is solid blue to that point, signifying a high recurrence of eye–hand coordination within this timeframe.

a) Exploration: an unsuccessful additive coordination Jan searches for a green where the distance between the bars remains relatively small until 06:01 (Fig. 6). Throughout this process, he discovers green in two abrupt instances. His main concern in this stage is understanding which bar should be higher.

Jan raises and lowers both fingers at the same rate throughout this stage, demonstrating an additive strategy. While doing so, his eyes mainly go back and forth between the top of

Time	Speaker	
03:45	Jan	*Hey! So one must be higher.*
03:46	Researcher	*One must be higher?*
03:48	Jan	*Yes.*
0350	Researcher	*And if you now remember that and start searching above a green* [location], *what can you do?*
03:55	Jan	*Leave one up and one down.*

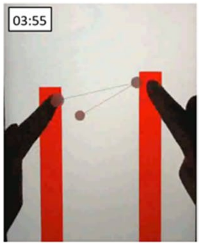

Fig. 6 Gaze transitions: L1 → R1 → L1

the bars (between L1 and R1, Fig. 6). This strategy is steady coordination that does not consider green feedback. The hands seem to lead the eye–hand interaction because there is no anticipatory eye-gaze movement.

b) Perturbation: the researcher offers a constraint Around the 6-min mark, Jan discovers a new green (Fig. 7). This time, he keeps his hands steady. The researcher asks Jan if he notices anything about the bars' position, and Jan responds that he knows now that the left bar should be lower than the right bar.

At the beginning of this interaction, Jan's eye-gaze keeps transitioning between the bars' tops. By asking Jan to "say how big the difference is," the researcher refers to the vertical distance between the two bars' heights. Jan responds to the researcher's cue by asking whether it is the size ("how big?") of the difference that the researcher is probing. Meanwhile, the RQA plot shows a decrease in blue-dot density, signifying gaze fluctuations, as Jan glances through AoIs that do not include his fingertips (e.g., M1 in Fig. 7).

The researcher clarifies that this ontology of "difference" is potentially a variable object, quality, or token whose magnitude in this context requires further specification—a "rule" Jan should figure out. He offers Jan an empirical pathway for deducing this rule from what he sees (in Dutch, Die zou je kunnen afleiden aan wat je nu ziet), directing him to find another green location at the top of the screen. Thus, he implies that investigating the difference at another location would promote Jan's establishing of the desired rule. From a CDS perspective, the researcher perturbs Jan's interaction with the technology to extract him from a basin of attraction (Ott, 2006), beckoning him toward other green pastures.

Time	Speaker	
07:06	Researcher	*Yes, and can you say something about how big the difference is?*
07:11	Jan	*Do you mean I have to say how big the difference is?*
07:43	Researcher	*[...] it is up to you to discover what that rule is. [...] you can deduce it from what you see. Can you think of something to get green at the top too?*
07:52	Jan	*Continue?*

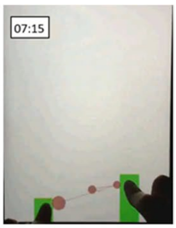

Fig. 7 Gaze transitions: L1 → R1 → M1

Time	Speaker	
09:11	Jan	*This one* [points at the right bar] *is always very long, and this one* [left bar] *is smaller.*
09:16	Researcher	*The right* [bar]?
09:16	Jan	*But also smaller. Yes.*
09:18	Jan	*Yes, the right one is very large, and the left one is getting bigger; the difference is getting smaller, I mean bigger.*

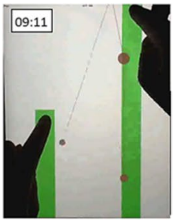

Fig. 8 Gaze transitions: L1→R0→R1→R3

c) Discovery: destabilization of eye-gaze patterns Jan continues to move his fingers close together, up and down (Fig. 8). His eye-gaze keeps moving between the bar tops, yet this time with more “stops” and visual searches at locations other than the two fingers.

Jan begins to self-impose a new constraint (“difference”) on his sensorimotor attempts to perform the task. The perturbation in the eye–hand system introduced by the tutor’s question about the difference leads to destabilizing the initial eye-gaze pattern that follows the fingers’ movement. As Jan discovers diverse green locations more consistently, more gaze pattern variants emerge. At the same time, Jan transitions from the changing magnitudes of either of the bars (“very long,” “smaller,” “very large,” and “bigger”) to the change in their differences (“the difference is getting smaller, I mean bigger”).

d) Discovery: stabilization of eye-gaze patterns (Fig. 9) Now, Jan reflects on prior green solutions and explains the change in the “difference” between the fingers’ positions when the hands are relatively low and relatively high. When the researcher asks him to describe that change, around 9:51, Jan’s gaze forms the pattern shown in Fig. 9, running from the bottom of the right bar to the top of the left to the middle of the right and up to the right finger. From this point on, eye movements will coalesce into steady patterns (Fig. 9), marking the beginning of the Fluency stage (Fig. 5 (d), (e)). At 09:59, Jan expresses the target proportion between the bars—half.

e) Fluency: coordination (Fig. 10) From this point on in the problem-solving, Jan demonstrates new eye–hand coordination that is based on a new visual structure—while both hands move upwards, the eye-gaze pattern moves between the top of the left bar, the middle of the right bar, and the top of the right bar (Fig. 10). An attentional anchor emerges.

Time	Speaker	
09:36	Jan	*It was just like that, when it was here* [moves the bars to the previous location], *it was a small difference, 3 centimeters or so, and here* [moves the bars up again], *it was something of 5 or so, 5, 6 centimeters. More, 10.*
09:51	Researcher	*Yes, and can you say something more about the difference between the two bars?*
09:59	Jan	*This one is completely filled all the time, more up, and this one* [left] *is about half.*

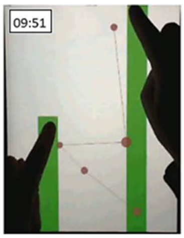

Fig. 9 Gaze transitions: R1→R2→L1→M2→M3

Time	Speaker	
10:05	Researcher	*Okay, and if you know that now* […], *put your fingers down at the bottom of the screen, and then move both of the bars up at the same time, not at the same height, and make sure that the bars stay green.*
10:20	Jan	[moves both fingers in green] *Stay green?*

Fig. 10 Gaze transitions: R2 → L1 → R2

5 Discussion

This study used constructs and methodologies from complex dynamic systems literature to analyze eye–hand dynamics as students solve embodied mathematics Trainer tasks. We demonstrated how students' changes in eye–hand coordination dynamics are consistent with the nonlinear phase transition characteristics of complex dynamic systems. We further showed how one students' eye–hand system shifted into a new dynamic equilibrium when solving the Trainer task.

To answer the first research question, we compared the RQA metrics of three typical stages in solving the Trainer task: Exploration (before discovering any green), Discovery (green locations search after prior discovery), and Fluency (hands move together in green). Across the 32 participants, the three stages were significantly different. The Fluency stage, where task-effective hand movements were relatively coordinated, coincided with more repetitive, predictable, and stable eye-gaze patterns respecting the manually changed AoI. That is, eye patterns guided hand-movement patterns, creating a complex dynamic system in equilibrium and performing a task-effective movement. Recurrence rate and percent determinism increased significantly in the Fluency stage compared to the earlier Exploration and Discovery stages, meaning that the participants' eyes–hands systems mostly reached high repetition and predictability at the Fluency stage. Mean line length and percent determinism were also highest during the Fluency stage, indicating higher stability and consistency. Across participants, then, gaze patterns became more dynamically stable and consistent as participants drew their hands into increasing coordination.

We chose a case study of a prototypical participant's mathematics learning with the Trainer to answer the second research question. We view Jan's case as demonstrating a cognitive system undergoing phase transition. It repelled out of a fixed-interval ineffective solution (pre-proportion), through an unpredictable and unstable low-coordination stage, into an effective proportional solution. The solution was constituted as eye-hand coordination dynamics. RQA metrics for Jan's learning process showed progress from stable yet functionally unsuccessful eye–hand dynamics, through a stage of instability, to ultimate task-effective eye–hand dynamic coordination. This coordination manifested as increases in recurrence rate, determinism, mean line, and trapping time during the Fluency stage. Thus, concerning earlier stages, Jan showed in the Fluency stage more repetitive, predictable, stable, and continuous intermodal (eyes–hands) and intramodal (hands–hands) dynamic coordination (see also Tancredi et al., 2022). Entropy, an indicator of the system's recurring coordination variability, also increased during the Fluency stage. Later in the discussion, we will discuss this finding (as relevant to all participants).

Further eye-tracking and video data analysis show that the transition to Fluency happened when Jan adopted new task-effective visual structures, consistently including the middle of the right bar (Figs. 8–10). Notably, before and within the Fluency stage, movements were centered on dynamically invariant visual orientations (attentional anchors, Abrahamson & Sánchez-García, 2016). While the final attentional anchor worked well to solve the task, Jan also manifested at the beginning a pre-intervention spontaneous attentional anchor focused either on the bars' or the fingers' heights. This ineffective attentional anchor may have stemmed from coupling the task description (move the bars) and a natural inclination of humans to look at their index fingers when they reach for objects and grasp them (Thulasiram et al., 2020). Congruent with enactivist theories that view cognitive development as the perceptual reorganization of action (Hutto et al., 2015; Varela et al., 1991), the eye-gaze joins the hands to provide a unified, embodied solution to the mathematics learning problem.

In solving the mathematics Trainer task, participants repelled from one attractor and precipitated towards another. RQA metrics showed low recurrence, predictability, stability, and consistency. Also, recurrence, predictability, and consistency had a lower variance in the Discovery phase than in the other two phases, implying that this decrease in RQA metrics was relatively consistent among all study participants, regardless of their performance in the Fluency stage. In this Discovery stage, the sociomaterial context, including the instrumented field of promoted action and the attentive cultural facilitator, played critical supporting roles in the emergence of new visual solutions (as in Flood et al., 2020; Shvarts & Abrahamson, 2019). The researcher orients Jan's visual attention to the difference between the bars and proposes he might deduce the rule from what he sees. Instructors can foster students' construction of goal-oriented visual structures mediating effective interactions. For example, Liao and Masters (2001) demonstrated that table tennis coaches could cause athletes to envisage imaginary constellations of dynamical environmental features to improve striking (and see Hutto & Sánchez-García, 2015, for the case of skateboard experts).

In Jan's case study, this is an iterative process in which the researcher nudges Jan to leave one basin of attraction and reach a new one (Ott, 2006). It is common to have more than one attractor in complex dynamic systems. For each such attractor (or desired state in each case), its basin of attraction is the set of initial conditions leading to the generation of long-term behavior that approaches that attractor. Thus, the qualitative behavior of the long-term motion of a given system can be fundamentally different depending on which basin of attraction the initial condition lies in. In this case study, we see Jan leaving one set of initial conditions for another.

Coordinating eye-gaze with hand movements became the problem of reducing the number of independent variables to control (Turvey, 1990). Right before the Fluency stage begins, we observe a drop in the recurrence rate (between Fig. 5 (c), (d)), and video data provide evidence for eye-gaze patterns to correct the bimanual solution (Figs. 8 and 9). Like the table tennis players in Liao and Masters (2001), Jan projected a new self-constraint onto the field of promoted action in order to enact a better-adapted coordinated movement (Fig. 7), manifested in our analyses as an increase in all RQA metrics (Fig. 5 and Table 2).

This study has several limitations as the first venture into RQA gaze analysis of embodied-mathematics learning. One technical limitation is the low sampling rate for eye-tracking data to match video data; this is relatively low for eye-tracking studies and does not allow monitoring of more detailed gaze. Furthermore, we were not able to access the exact eye-tracking accuracy for this study. However, eye-tracking accuracy for the particular eye-tracking model that we have used in our study (Tobii X2-30) has been reported to be sufficient for the purposes of our study. Further research with greater frequency and eye-tracking accuracy may capture the full, nuanced dynamics of gaze development. A population-sampling limitation is that this study only includes participants who spent significant time in all three stages of learning predefined by our judgment of a solution process, omitting those who did not exhibit one of them. Further research on the dynamics of those participants would support a more nuanced understanding of learners' diverse intrinsic dynamics. An eye-tracking sampling limitation (10 Hz in the case of the current study) does not affect the methodologies applied in this paper but still bounds the empirical insights provided in this study since eye-tracking studies in mathematics education often consider 30 Hz as a threshold sampling rate (Strohmaier, 2020). However, our interest in this study is related to visual attention relative to the hands' movement rather than raw gaze patterns, where gaze data is categorically coded. The significant statistical effects imply that the phenomena presented in this study can be modeled and explained in slightly larger grain sizes than typical eye-tracking studies in mathematics education.

Nonetheless, further studies can handle this lacuna, controlling for inconsistent sampling rates between different data collection sources. A software limitation is that the Trainer task does not feature an additive stable phase. Thus, although we observed in many participants evidence of additive movement, the stability of additive movement as a prior attractor was not solicited by us and was not consistently observable in these data. Finally, from a theoretical standpoint, this analysis examines the relationship between hand and gaze dynamics at the level of meta-stages in bimanual coordination. Coordination among a system's components also refers to different degrees of functional order among interacting parts through space and time.

Future research can characterize the evolution of this coordination through analyses using a smaller grain size to chart how hand and gaze enter into coordination (Tancredi et al., 2022). A smaller grain size would also help elucidate entropy dynamics in these data. We found entropy to decrease during Discovery relative to Exploration and Fluency—an increase and subsequent decrease in the level of order in the gaze system. This finding contradicts the spike and consequent drop in entropy predictive of phase transitions in the dynamic systems literature (as in Stephen & Dixon, 2009). In stage-based analyses here and in Tancredi et al. (2021), we could not detect such a pattern. One explanation for these findings could be that the Fluency stage was signified by an increased recurrence rate and mean line length. Since entropy represents the level of disorder in line lengths in the RQA plot, more recurrent cases may be more prone to more significant variance/disorder.

6 Conclusions

This study presents a nascent contribution to mathematics education research, demonstrating mathematics learning as a complex dynamic system in flux. Our findings advance enactivist research on mathematics education. Pirie and Kieren (1989) applied enactivist principles to implicate mathematical cognition as grounded in dynamic images. Nemirovsky et al. (2013) viewed mathematical learning as sensorimotor coordination. Abrahamson et al. (2016) offered a qualitative demonstration of enactivist mathematics learning, corroborating Piaget's systemic notion of reflecting abstraction. Abrahamson and Trninic (2015) put forth the thesis that the theory and methods of coordination dynamics can furnish explanatory models of mathematical cognition in flux, and Abrahamson and Bakker (2016) argued for the importance of exploration and discovery in grounding fluency with new mathematical movements (see also Abrahamson, 2018; Abrahamson & Abdu, 2020). Whereas these principles and methodologies of complex dynamic systems have been previously employed to model cognitive development (Thelen & Smith, 1994) and problem-solving (Stephen & Dixon, 2009), we have shown the emergence of hand–hand (Tancredi et al., 2021) and hand–gaze coordination in our enactivist tasks. To the best of our knowledge, this is the first study to offer quantitative empirical data as evidence supporting the thesis that mathematical learning is a nonlinear process of sensorimotor coordination (see also Tancredi et al., 2022). In particular, our findings highlight the pivotal constitutive role of developing new visual orientations in organizing the enactment of challenging movements (Mechsner, 2003; Mechsner et al., 2001). As such, we spotlight visual perception-for-action as a central phenomenon of interest in mathematical cognition and learning research.

We argue that eye–hand coordination is the cognitive vehicle of enactive mathematical learning. Grounding a new mathematical concept is developing the capacity to enact visually guided actions. These actions instantiate the concept. Thus, the Mathematics Imagery Trainer implements pedagogical principles of radical constructivism (Steffe & Kieren, 1994) in the form of an instrumented field of promoted action (Abrahamson & Trninic, 2015), fostering concept-grounding schemes. The larger research program harnesses enactivist philosophy (Petitmengin, 2007) in theorizing the phenomenology of mathematical learning as literally coming to grasp new structures (Abrahamson, 2021). RQA can serve this research program by modeling dialogic interactions (tutor–student or student–student) regarding independent complex systems reaching coordination.

By investigating the enactive roots of attentional anchors for mathematical concepts, we could all stand to better engineer action-based embodied designs that serve broad population sectors, optimizing for students' diverse sensory and motor capabilities (for inclusive designs, see Abrahamson et al., 2019; Lambert et al., 2022). Embodied design technologies for mathematics learning generate useful empirical contexts for studying the microgenesis of multimodal conceptual development. We envision this design-based research program as harnessing state-of-the-art methodological techniques for real-time multimodal learning analytics to build artificially intelligent, responsive tutors (Abdullah et al., 2017; Pardos et al., 2018). In so doing, Trainers could continue serving the field as an auspicious empirical platform for conducting design-based research into enactivist models of mathematics education (Hutto, 2019; Hutto et al., 2015). Though still in the early days of research, development, and dissemination, Trainers offer mathematics education a new activity genre. Trainers are grounded in long-standing learning theory and suited to timely technological development and social circumstances, such as remote learning in the age of global pandemics (Shvarts & van Helden, 2021).

Data availability The data that support the findings of this study are available from the Freudenthal Institute at Utrecht University, but restrictions apply to the availability of these data, which were used under license for the current study and so are not publicly available. However, data are available from the authors upon reasonable request and with the permission of the Freudenthal Institute at Utrecht University.

Declarations

Conflict of interest The authors declare no competing interests.

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